Determinants of finite dimensional algebras

Matthieu Romagny

Abstract: To each associative unitary finite-dimensional algebra over a normal base, we associate a canonical multiplicative function called its *determinant*. We give various properties of this construction, as well as applications to the topology of the moduli stack of n-dimensional algebras.

1 Introduction

Our object of interest in this article is the moduli stack of n-dimensional associative algebras with unit, denoted $\mathfrak{Alg}_n$. Given a free module with basis $e_1, \dots, e_n$, an algebra structure is given by the constants c_k^{ij} such that $e_ie_j = \sum_k c_k^{ij} e_k$, satisfying the relations coming from the associativity rule $(e_ie_j)e_k = e_i(e_je_k)$. Therefore, algebras with a given basis are classified by the affine variety of structure constants c_k^{ij} cut out by these relations, denoted Alg_n. There is an action of the group G of base changes, and $\mathfrak{Alg}_n$ is the quotient stack of Alg_n by this action. The geometry of Alg_n remains rather mysterious, and the attention of the specialists has focused on topics like the determination of the number of irreducible components (open question, first raised by Gabriel in [Ga]), asymptotic bounds for their dimensions, and the existence of smooth components ([LBR]) or nonreduced components ([DS]). For more details and references, we refer to [LBR] and [P].

In this paper, we introduce a chain of closed substacks

$$\mathfrak{Alg}_{n,\le 2} \subset \mathfrak{Alg}_{n,\le 3} \subset \dots \subset \mathfrak{Alg}_{n,\le n} = \mathfrak{Alg}_n$$

indexed by the *degree of algebraicity* d. Roughly, an algebra is in $\mathfrak{Alg}_{n,\le d}$ if all its monogenic subalgebras have rank less than d. For example, by the Cayley-Hamilton theorem, the algebra of (n, n) matrices lies in $\mathfrak{Alg}_{n^2,\le n}$. Then, we construct a *determinant* on the normalization of the reduced substack of $\mathfrak{Alg}_{n,\le d} \setminus \mathfrak{Alg}_{n,\le d-1}$, by which we mean a function enjoying the properties listed in the result below which is our main theorem.

Before we state it, a word about the terminology is necessary. Any locally free algebra over a ring R (or locally free sheaf of algebras over $S = \mathrm{Spec}(R)$) may be seen as a vector bundle over S endowed with an algebra structure over the ring scheme $\mathbb{G}_{a,S}$ (*cf* section 2). We use the point of view of algebra schemes because most identities that we prove (*e.g.* multiplicativity of the determinant) are polynomial identities and not just equalities of functions. Also this choice makes our statements at the same time more precise and more elegant. However, the reader unfamiliar with the vocabulary of schemes may very well replace all "algebra schemes over a scheme S" by "(ordinary) algebras over a ring R". In the text, a special effort is made to translate into commutative algebra the statements with a strong flavour of algebraic geometry, wherever it seems necessary. Now here is our main result :

Theorem (4.1.1) *Let S be a normal integral scheme, A/S an n-dimensional $\mathbb{G}_{a,S}$-algebra scheme with $n \ge 2$, $A^\vee/S$ its linear dual, and $d \ge 2$ the least integer such that A/S belongs to $\mathfrak{Alg}_{n,\le d}$. Then there exists a unique section of* $\mathrm{Sym}^d(A^\vee)$ *denoted* $\det_{A/S}$ *or simply* $\det$*, such that :*

(1) $\det\colon A \to \mathbb{G}_{a,S}$ *is a morphism of multiplicative unitary monoid schemes, and*

(2) *the morphism of schemes $A \to A$ defined by evaluating on $a \in A$ the polynomial $P(T) = \det(T-a)$ is zero. (Here T is a scalar in the $\mathbb{G}_{a,S}[T]$-algebra $A \times_S \mathbb{G}_{a,S}[T]$.)*

The determinant satisfies the further properties :

Date : December 4, 2007.
2000 Mathematics Subject Classification : Primary 16G10, Secondary 14D20.

(3) *the group scheme of units of A is the preimage of the multiplicative group $\mathbb{G}_{m,S}$ under* det *;*

(4) *the formation of* det *is compatible with flat extensions of normal integral schemes $S' \to S$.*

The existence of the determinant has some consequences on the topology of $\mathfrak{Alg}_n$. In fact we hope that it helps to sort out the irreducible components (see subsection 5.3 and in particular question 5.3.3). As an example, the irreducible component of $\mathrm{M}_n(k)$ has received special attention due to its connection with moduli spaces of vector bundles on curves (see [LBR]). The following proposition, proved using the determinant, implies that it is contained in $\mathfrak{Alg}_{n^2,\leq n^2-n}$:

Proposition (5.3.2) Let $n_1, \dots, n_r \geq 1$ be integers and $n = (n_1)^2 + \dots + (n_r)^2$. Assume that one of the n_i is at least 2 and denote by ν their infimum. Then the irreducible component of the algebra $\mathscr{A}_0 = \mathrm{M}_{n_1}(k) \times \dots \times \mathrm{M}_{n_r}(k)$ is contained in $\mathfrak{Alg}_{n,\leq n-\nu}$.

Let us finally survey the contents of the paper. In section 2, we explain the correspondance between algebras, or sheaves of algebras, and algebra schemes. We include the case of infinite dimension to incorporate the polynomial algebra $\mathbb{G}_{a,S}[T]$ that appears in the main theorem. In section 3 we define the degree of algebraicity and the closed substacks $\mathfrak{Alg}_{n,\leq d}$, and we give their basic properties. In section 4 we prove the main result (theorem 4.1.1) and we use it to define the determinant on the normalization of the reduced substack of $\mathfrak{Alg}_{n,\leq d} \setminus \mathfrak{Alg}_{n,\leq d-1}$ (proposition 4.1.2 ; we conjecture that it is actually unnecessary to normalize). We also prove that the determinant is invariant under all (anti)automorphisms of the algebra (proposition 4.2.1). In section 5 we provide basic computations of determinants : determinant of the opposite algebra (5.1.1), determinant of a product (5.1.2). Also we use the determinant to define intrinsic invariants of an algebra : trace, discriminant, unimodular group (5.2) and to study the topology of $\mathfrak{Alg}_n$ (5.3). Finally in section 6 we compute more examples of determinants : field extensions, quaternion algebras, exterior algebras, and three-dimensional algebras.

Contents

2 Algebras and algebra schemes

In this paper, all algebras are unitary and associative, but not necessarily commutative.

2.1 Different ways to see an algebra

2.1.1 Let S be a scheme. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_S$-module, we denote the dual module by $\mathcal{F}^\vee$. Associated to $\mathcal{F}$ are the *tensor algebra* $\mathrm{T}(\mathcal{F})$, the *symmetric algebra* $\mathrm{Sym}(\mathcal{F})$, and the *exterior algebra* $\wedge\mathcal{F}$. If $\mathcal{F} = (\mathcal{O}_S)^n$ for some integer n, and we denote by $U_1, \dots, U_n$ its canonical basis, then $\mathrm{T}(\mathcal{F})$ is the noncommutative polynomial algebra $\mathcal{O}_S\{U_1, \dots, U_n\}$ and $\mathrm{Sym}(\mathcal{F})$ is the commutative polynomial algebra $\mathcal{O}_S[U_1, \dots, U_n]$. More generally, if $\mathcal{F}$ is locally free of finite rank, then $\mathrm{T}(\mathcal{F})$ and $\mathrm{Sym}(\mathcal{F})$ are twisted (non)commutative polynomial algebras.

Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_S$-algebra and $a = (a_1, \dots, a_n)$ a tuple of sections of $\mathcal{A}$ over S. The morphism of *evaluation at* a is the morphism of algebras

$$\mathrm{ev}_a \colon \mathcal{O}_S\{U_1, \dots, U_n\} \to \mathcal{A}$$

(or $\mathrm{ev}_a \colon \mathcal{O}_S[U_1, \dots, U_n] \to \mathcal{A}$ if the sections a_i commute) defined by $P(U_1, \dots, U_n) \mapsto P(a_1, \dots, a_n)$. Alternatively, in defining these notions, one can exhibit a more canonical domain. Namely, one can define $\mathcal{F}$ to be the sub-$\mathcal{O}_S$-module of $\mathcal{A}$ generated by $a_1, \dots, a_n$, and consider the morphism of algebras $\mathrm{T}(\mathcal{F}) \to \mathcal{A}$, resp. $\mathrm{Sym}(\mathcal{F}) \to \mathcal{A}$, induced by the inclusion $\mathcal{F} \to \mathcal{A}$. This map may again be called *evaluation at* a ; note that $\mathrm{T}(\mathcal{F})$, resp. $\mathrm{Sym}(\mathcal{F})$, is indeed a polynomial algebra only if $\mathcal{F}$ is free. Finally we say that a *generates* $\mathcal{A}$ if ev_a is surjective.

2.1.2 It is sometimes possible to view an $\mathcal{O}_S$-module or $\mathcal{O}_S$-algebra as a scheme. In order to explain this, we need to recall that, just in the same way as one defines the notion of a group scheme, we have the notion of a ring scheme. The most important example of a commutative ring scheme is the *structure ring scheme* denoted $\mathbb{G}_{a,S}$, or simply $\mathbb{G}_a$ if no confusion seems likely to result. Its underlying scheme is the affine line over S, and it has two composition laws $\mathbb{G}_a \times_S \mathbb{G}_a \to \mathbb{G}_a$ called addition and multiplication, and two sections $S \to \mathbb{G}_a$ denoted $0, 1$ which are neutral for these laws.

In the same way as one identifies an S-scheme with its functor of points, we will identify an $\mathcal{O}_S$-algebra (or an $\mathcal{O}_S$-module) with its functor of sections. This is made possible by the embedding $\mathcal{S}ec \colon (\mathcal{O}_S\text{- Alg}) \to \mathrm{Funct}((\mathrm{Sch}/S)^\circ, \mathbb{G}_{a,S}\text{- Alg})$ defined as follows : the functor $\mathcal{S}ec$ takes $\mathcal{A}$ to the functor $\mathcal{S}ec_{\mathcal{A}}$ defined by

$$\mathcal{S}ec_{\mathcal{A}}(T \xrightarrow{f} S) = \text{ the } \mathrm{H}^0(T, \mathcal{O}_T)\text{-algebra } \mathrm{H}^0(T, f^*\mathcal{A}) \ .$$

The same works for modules.

The functor of sections of an $\mathcal{O}_S$-algebra $\mathcal{A}$ whose underlying $\mathcal{O}_S$-module is locally free of finite rank n is (representable by) a $\mathbb{G}_{a,S}$-algebra scheme whose underlying $\mathbb{G}_{a,S}$-algebra module is an n-dimensional vector bundle. Indeed, it is representable by $A = \mathrm{Spec}(\mathrm{Sym}(\mathcal{A}^\vee))$. Concretely, if $U = \mathrm{Spec}(R)$ is an open affine subscheme of S such that $\mathcal{A}(U)$ is free, then given a basis $e_1, \dots, e_n$ of $\mathcal{A}(U)$, there are some constants $c_k^{i,j} \in R$ such that $e_i e_j = \sum_k c_k^{i,j} e_k$. The forms $t_i = e_i^*$ are a system of coordinates for A, that is to say $A = \mathrm{Spec}(R[t_1, \dots, t_n])$, and the multiplication of A is given, on the level of functions, by $t_k \mapsto \sum_{i,j} c_k^{i,j} t_i \otimes t_j$. For simplicity, in all the paper we will call such an algebra an *n-dimensional algebra*.

We take the opportunity here to mention that our conventions for vector bundles differ slightly from the ones in [EGA], in that we will use the words *vector bundle* only for those bundles whose sheaf of sections is locally free. Also, if $\mathcal{F}$ is a locally free module of finite rank on a scheme S, then the *associated vector bundle* will be $F = \mathrm{Spec}(\mathrm{Sym}(\mathcal{F}^\vee))$ and not $\mathrm{Spec}(\mathrm{Sym}(\mathcal{F}))$.

More generally, an $\mathcal{O}_S$-algebra whose underlying module is locally free of arbitrary rank is mapped to an inductive limit of finite-dimensional vector bundles over S ; in the inductive system, the morphisms are vector bundle homomorphisms. Examples include the polynomial algebras $\mathbb{G}_{a,S}\{U_1, \dots, U_n\}$ and $\mathbb{G}_{a,S}[U_1, \dots, U_n]$ associated to the polynomial algebra sheaves $\mathcal{O}_S\{U_1, \dots, U_n\}$ and $\mathcal{O}_S[U_1, \dots, U_n]$.

2.2 Universal elements

In order to establish some identities concerning algebra schemes, it will often be enough to check these identities for their *universal element*. Let us indicate briefly the relevant notations and terminology.

2.2.1 Let $A \to S$ be a morphism of schemes. The *universal element* of A/S is the section of the pullback scheme $\mathrm{pr}_2 \colon A \times_S A \to A$ which is given by the diagonal of A/S. It is denoted by the letter α. If $f \colon A \to B$ is a morphism of schemes over S, then $(f \times \mathrm{id}_A) \circ \alpha = f^*\beta$ where α and β are the universal elements of A and B. If f is scheme-theoretically dominant, that is to say if the scheme-theoretic image of f is B, then this means that the universal element of A maps to the universal element of B.

2.2.2 For local computations, when A is an n-dimensional $\mathbb{G}_{a,S}$-algebra scheme and S is the spectrum of a ring R, it is useful to have a description in terms of commutative algebra. In this case A is determined by its algebra of global sections $\mathscr{A} := A(R)$. We denote by $R_{\mathscr{A}} := \mathrm{Sym}(\mathscr{A}^\vee)$ its function ring, which is a graded R-algebra ; here $\mathscr{A}^\vee$ is the dual R-module of $\mathscr{A}$. Because $\mathscr{A}$ is projective of finite rank, there is an isomorphism of R-modules $\mathrm{Hom}_R(\mathscr{A}, \mathscr{A}) \simeq \mathscr{A} \otimes_R \mathscr{A}^\vee$, and since $R_{\mathscr{A}}$ contains $\mathscr{A}^\vee$ as the piece of degree 1, the target module is a submodule of $\mathscr{A} \otimes_R R_{\mathscr{A}}$. In this setting, the universal element α is the image of the identity $\mathrm{id} \in \mathrm{Hom}_R(\mathscr{A}, \mathscr{A})$ inside $\mathscr{A} \otimes_R R_{\mathscr{A}}$. If $\mathscr{A}$ is free as an R-module, and $e_1, \dots, e_n$ is a basis, then $R_{\mathscr{A}}$ is the (commutative) polynomial R-algebra on the independant variables $t_i = e_i^*$ and α is just $t_1 e_1 + \cdots + t_n e_n$.

Now let A, B, C be finite-dimensional $\mathbb{G}_{a,S}$-algebra schemes over $S = \mathrm{Spec}(R)$, with sets of global sections $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and function rings $R_{\mathscr{A}}$, $R_{\mathscr{B}}$, $R_{\mathscr{C}}$.

If $f : A \to B$ is a surjective homomorphism, there is an associated map $\mathscr{A} \to \mathscr{B}$ and an injection $R_{\mathscr{B}} \hookrightarrow R_{\mathscr{A}}$. It is easy to check that the image of the universal element $\alpha \in A \otimes_R R_{\mathscr{A}}$ under the map $f \otimes \mathrm{id} \colon A \otimes_R R_{\mathscr{A}} \to B \otimes R_{\mathscr{A}}$ is the universal element $\beta \in B \otimes_R R_{\mathscr{B}} \hookrightarrow B \otimes_R R_{\mathscr{A}}$. Indeed, this assertion is local on R, so we may localize and choose a basis $e_1, \dots, e_n$ of A such that f is the projection onto the subspace spanned by the first k vectors $e_1, \dots, e_k$. Then $\alpha = \sum_{i=1}^n e_i^* e_i$ and $f(\alpha) = \sum_{i=1}^k e_i^* e_i = \beta$.

In particular, a product algebra $C = A \times B$ has function ring $R_{\mathscr{C}} = R_{\mathscr{A}} \otimes_R R_{\mathscr{B}}$, with universal element $\gamma = (\gamma_1, \gamma_2) \in \mathscr{C} \otimes_R R_{\mathscr{C}}$. By the above, the projection γ_1 in $\mathscr{A} \otimes_R R_{\mathscr{C}}$ is $\alpha \in \mathscr{A} \otimes_R R_{\mathscr{A}} \subset \mathscr{A} \otimes_R R_{\mathscr{C}}$, and the projection γ_2 in $\mathscr{B} \otimes_R R_{\mathscr{C}}$ is β. We point out that if $A = B$, it is not true that $\gamma_1 = \gamma_2$, because $R_{\mathscr{A}} = R_{\mathscr{B}}$ is embedded in two different ways in $R_{\mathscr{C}}$.

3 Degree of algebraicity

3.1 Definition

Let $A \to S$ be an n-dimensional $\mathbb{G}_{a,S}$-algebra scheme. We say that *the degree of algebraicity of A/S is less than or equal to d*, written $\deg(A/S) \le d$, if the morphism :

$$\begin{array}{rcl} A & \to & \wedge^{d+1} A \\ a & \mapsto & 1 \wedge a \wedge \cdots \wedge a^d \end{array}$$

is the zero morphism. This is the same thing as saying that the universal element α satisfies $1 \wedge \alpha \wedge \cdots \wedge \alpha^d = 0$. Locally on S, we may choose a basis for A, and then the latter condition is equivalent to the vanishing of the minors of size $d+1$ of the matrix whose columns are the powers of α. If S is the spectrum of a field k, then we say that *the degree of algebraicity of A/k is equal to d*, written $\deg(A/k) = d$, if it is less than d but not less than $d-1$. If S is arbitrary, we say that $\deg(A/S) = d$ if $\deg(A/S) \le d$ and all the fibres of $A \to S$ have degree of algebraicity equal to d. If A/S has a degree of algebraicity d, then $A \times_S S'$ has degree of algebraicity d for all extensions $S' \to S$. Also if A has a degree of algebraicity d, then $d \le n$ because $\wedge^{n+1} A = 0$. To better illustrate this definition, consider the following example.

Example 3.1.1 Let $R = k[\epsilon]/\epsilon^2$ be the ring of dual numbers over a field k. Consider the 3-dimensional commutative algebra $\mathscr{A} = R[x,y]/(x^2 - \epsilon x, xy, y^2)$ and its associated R-algebra scheme A. If we denote the universal element by $\alpha = r + sx + ty$, we have $1 \wedge \alpha \wedge \alpha^2 = -\epsilon s^2 t \cdot 1 \wedge x \wedge y$. This is zero modulo ϵ, so $\deg(A \otimes k) = 2$, but we do not attribute a degree of algebraicity to A itself.

3.2 Relation with monogenic subalgebras

If k is a field, then any element x in a finite-dimensional (ordinary) k-algebra has a minimum polynomial, whose degree is also the dimension of the subalgebra generated by x ; we call it the *degree of* x, denoted $\deg(x)$. The degree of algebraicity of an algebra as defined above has a simple meaning in these terms :

Lemma 3.2.1 *Let A/S be an n-dimensional algebra scheme that has a degree of algebraicity d.*

(1) *If S is the spectrum of a field k, there is a finite separable field extension k'/k such that $d = \sup\{\deg(x),\ x \in A(k')\}$. If k is infinite we may take $k' = k$.*

(2) *We have $d = n$ if and only if there exists a surjective étale extension $S' \to S$ such that $A \otimes_S S'$ is monogenic. If S has infinite residue fields, we may take for S' a Zariski open cover of S. In particular, if $d = n$ then A is commutative.*

Proof : (1) Choose a basis $e_1, \dots, e_n$ for $\mathscr{A} = A(k)$ and let $t_i = e_i^*$, so $\alpha = t_1 e_1 + \dots + t_n e_n$. Also let M_i denote the matrix with columns $1, \alpha, \dots, \alpha^i$. To say that $d = \deg(A)$ means that all the minors of size $d+1$ of M_d vanish, and one of the minors of size d of M_{d-1} does not vanish, call $m \in k[t_1, \dots, t_n]$ this minor. Thus, clearly, all elements of $\mathscr{A}$ have degree less than d, since they are specializations of α. Now if k is infinite, there is a tuple of elements $(x_1, \dots, x_n) \in k^n$ where m takes a nonzero value, so $x = x_1 e_1 + \dots + x_n e_n$ has degree $\deg(x) = d$. If k is finite there is such a tuple in an algebraic closure of k, hence in a finite separable extension k'/k.
(2) The *if* part is obvious, we focus on the *only if*. The claim is local on S so we may assume that $S = \mathrm{Spec}(R)$ is affine and A is trivial as a vector bundle, determined by $\mathscr{A} = A(R)$. Let m be a maximal ideal of R and $k = R/m$. By point (1) there is a finite separable field extension k'/k such that $\mathscr{A} \otimes_R k'$ is monogenic, and if k is infinite we may take $k' = k$. Let $P \in R[T]$ be a monic polynomial with reduction $p \in k[T]$ such that $k' = k[T]/(p)$, and let $R' = R[T]/(P)$. Since R'/R is finite flat and étale at m, we may localize in R' if necessary and assume that R'/R is étale. Now let $g \in \mathscr{A} \otimes_R R'$ be a lift of a generator of $\mathscr{A} \otimes_k k'$. By Nakayama's lemma, after a further localization in R', g is a generator of $\mathscr{A} \otimes_R R'$. The last statement about commutativity follows immediately. □

Example 3.2.2 Consider the commutative $\mathbb{F}_2$-algebra $\mathscr{A} = \mathbb{F}_2[x,y]/(x^2 - x, xy, y^2 - y)$ and its associated $\mathbb{F}_2$-algebra scheme A. Then $\mathscr{A}$ satisfies $x^2 = x$ for all $x \in \mathscr{A}$, however its degree is not 2 because the morphism

$$\begin{array}{ccc} A & \to & \wedge^3 A \\ a & \mapsto & 1 \wedge a \wedge a^2 \end{array}$$

is $a = u + vx + wy \mapsto (vw^2 - v^2 w)(1 \wedge x \wedge y)$ which is not zero. In fact $\mathscr{A}$ has degree 3 and indeed $\mathscr{A} \otimes \mathbb{F}_4$ is generated as an algebra by $ux + y$ where $u \in \mathbb{F}_4$ is a primitive cubic root of unity.

3.3 Definition of $\mathfrak{Alg}_{n,\le d}$ and $\mathfrak{Alg}_{n,d}$

Given a basis, a unitary algebra structure is determined by the coefficients $c_k^{i,j}$ for all $2 \le i,j \le n$ and $1 \le k \le n$, such that $e_i e_j = \sum_k c_k^{i,j} e_k$, subject to the relations obtained by expansion of the associativity equations $(e_i e_j) e_k = e_i (e_j e_k)$. These relations tell us that the moduli scheme Alg_n is a closed subscheme of affine space of dimension $n(n-1)^2$ over $\mathbb{Z}$. The change of basis is expressed by

an action of the subgroup $G \subset \mathrm{GL}_n$ which is the stabilizer of the first basis vector, and $\mathfrak{Alg}_n$ is the quotient stack of Alg_n by G.

We denote by $\mathfrak{Alg}_{n,\leq d}$, resp. $\mathrm{Alg}_{n,\leq d}$, the closed substack, resp. the closed subscheme classifying n-dimensional algebras with degree of algebraicity $\leq d$. We introduce also the locally open substacks

$$\mathfrak{Alg}_{n,d} = \mathfrak{Alg}_{n,\leq d} \setminus \mathfrak{Alg}_{n,\leq d-1} \qquad \text{and} \qquad \mathrm{Alg}_{n,d} = \mathrm{Alg}_{n,\leq d} \setminus \mathrm{Alg}_{n,\leq d-1} \ .$$

Our main goal in the next section is to construct determinant functions on the normalized strata $\mathfrak{Alg}^{\sim}_{n,d}$, by which we mean, the normalization of the reduced stack $(\mathfrak{Alg}_{n,d})_{\mathrm{red}}$. We observe that $\mathfrak{Alg}^{\sim}_{n,d}$ is the quotient of $\mathrm{Alg}^{\sim}_{n,d}$ by G, because G is smooth and normality is local in the smooth topology.

4 Determinants

4.1 Construction of determinants

Now comes our main result :

Theorem 4.1.1 *Let S be a normal integral scheme, A an n-dimensional $\mathbb{G}_{a,S}$-algebra scheme, and $d \geq 2$ the least integer such that the morphism $A \to \wedge^{d+1} A$ defined by $a \mapsto 1 \wedge a \wedge \cdots \wedge a^d$ is zero. Then there exists a unique section of* $\mathrm{Sym}^d(A^\vee)$*, called the* determinant of A *and denoted* $\det_{A/S}$ *or simply* $\det$*, such that :*

(1) $\det \colon A \to \mathbb{G}_{a,S}$ *is a morphism of multiplicative unitary monoid schemes, and*

(2) *the morphism of schemes $A \to A$ defined by evaluating on $a \in A$ the polynomial $P(T) = \det(T - a)$ is zero. (Here T is a scalar in the $\mathbb{G}_{a,S}[T]$-algebra $A \times_S \mathbb{G}_{a,S}[T]$.)*

The determinant satisfies the further properties :

(3) *the group scheme of units of A is the preimage of the multiplicative group $\mathbb{G}_{m,S}$ under* $\det$ *;*

(4) *the formation of* $\det$ *is compatible with flat extensions of normal integral schemes $S' \to S$.*

Here are a few comments before we pass to the proof. One strong point of the theorem is that d is the degree of the generic fibre of A/S, not of all fibres. If we want to extend it to more general base schemes, we can expect such a strong result only for integral bases. An extension to arbitrary base schemes may be reasonable if one assumes really $\deg(A/S) = d$ in the sense of 3.1 (i.e. all fibres of A/S have degree d). Intuitively, the basic idea is to produce the determinant as the constant coefficient of the minimum polynomial of a generic element of the algebra, just like in the case of a matrix algebra. More precisely, we consider the universal element α of A, and we study the kernel of the evaluation morphism $\mathrm{ev}_\alpha \colon \mathbb{G}_{a,A}[T] \to A_A$. The left regular representation provides an embedding of A_A into $\mathrm{End}_A(A_A)$, its algebra of module endomorphisms. Hence α is cancelled by its Cayley-Hamilton polynomial CH_α. This provides a canonical polynomial of degree n in the kernel of ev_α. Over a normal integral scheme, we prove that the kernel of ev_α is generated by a single monic polynomial. Over a general base scheme S, this is not to be expected : for instance, in example 3.1.1 two generators are required, namely CH_α and the polynomial $P(T) = \epsilon(T - r)^2$, where we have written $\alpha = r + sx + ty$. So this raises the problem of finding a "distinguished" polynomial in $\ker(\mathrm{ev}_\alpha)$. Ideally it should be monic, it should divide CH_α and have minimal degree.

We come to the proof of theorem 4.1.1. Below, whenever we reduce to the local case, we will use the following notation : $S = \mathrm{Spec}(R)$ is affine and small enough so that the algebra of sections $\mathscr{A} = A(R)$ is free over R ; the function ring of A is $R_{\mathscr{A}} = \mathrm{Sym}(\mathscr{A}^\vee)$; the universal element is $\alpha \in \mathscr{A} \otimes_R R_{\mathscr{A}}$. We denote by K resp. $K_{\mathscr{A}}$ the fraction field of R resp. $R_{\mathscr{A}}$.

Proof of unicity : Normality is not useful here ; it is enough to assume that S is integral. If S is irreducible, the integer d defined in the statement of the theorem is the same for all open subschemes of S, hence proving unicity is a local question and we may assume that S is affine. We use the above notations for the affine case. Let $P \in K_{\mathscr{A}}[T]$ be the minimum polynomial of the universal element α. Clearly its degree is d. Consider the polynomial $Q(T) = \det_A(T - \alpha)$. Since the determinant is required to take 1 to 1, and is homogeneous of degree d, it follows that Q is nonzero and monic. By property (2) we have $Q(\alpha) = 0$, hence we get $Q = P$. Therefore $\det_A(\alpha) = (-1)^d P(0)$ and this determines $\det_A$. □

We add that a flat extension of normal integral rings $R \to R'$ preserves the kernels, hence the minimum polynomial of the universal element. Statement (4) of the theorem follows easily, by immediate globalization. It remains to prove that there exists a section $\det_{A/S}$ with the properties (1)-(2)-(3).

Proof of existence : By unicity, the question of existence is local and we may assume that S is affine and $\mathscr{A} = A(R)$ is free over R. Since the minimum polynomial P divides the Cayley-Hamilton polynomial CH_α, its roots (in some algebraic closure of $K_{\mathscr{A}}$) are roots of CH_α, hence integral over $R_{\mathscr{A}}$. Therefore the coefficients of P are themselves integral, and since they lie in $K_{\mathscr{A}}$ and $R_{\mathscr{A}}$ is normal, it follows that the coefficients of P are in $R_{\mathscr{A}}$. We define $\det = \det_A := (-1)^d P(0)$. This is an element in the graded algebra $R_{\mathscr{A}}$, and we now prove that it lies in the degree d component. In order to do so we fix a basis $e_1 = 1, e_2, \dots, e_n$ for $\mathscr{A}$. Let $t_i = e_i^*$ and write $\alpha = t_1 e_1 + \dots + t_n e_n$. Write the minimum polynomial of α as $P(t_1, \dots, t_n, T)$, so that by definition

$$P(t_1, \dots, t_n, t_1 e_1 + \dots + t_n e_n) = 0 \; .$$

Let X be an indeterminate. If we substitute Xt_i to t_i we find

$$P(Xt_1, \dots, Xt_n, Xt_1 e_1 + \dots + Xt_n e_n) = 0 \; ,$$

thus $P(Xt_1, \dots, Xt_n, XT)$ cancels α. By unicity of the minimum polynomial in $K_{\mathscr{A}}(X)[T]$, it follows that $P(Xt_1, \dots, Xt_n, XT) = X^d P(t_1, \dots, t_n, T)$. Therefore P is homogeneous of degree d in $t_1, \dots, t_n, T$ and hence det is homogeneous of degree d in $t_1, \dots, t_n$.

We proceed to prove (1). For this we extend the base to the normal domain $R' := R_{\mathscr{A} \times \mathscr{A}} = R_{\mathscr{A}} \otimes_R R_{\mathscr{A}}$, and we have to prove that $\det(\alpha_1 \alpha_2) = \det(\alpha_1) \det(\alpha_2)$ where (α_1, α_2) is the universal element of $\mathscr{A} \times \mathscr{A}$. To prove this identity, we may embed $\mathscr{A} \otimes_R R'$ into $\mathscr{A} \otimes_R \overline{K}'$, where $\overline{K}'$ is an algebraic closure of the fraction field of R'. For simplicity we will write K for $\overline{K}'$ and $\mathscr{A}$ for $\mathscr{A} \otimes_R \overline{K}'$. Now let $V_0 = 0 \subset V_1 \subset V_2 \subset \dots \subset V_d = A$ be a composition series for the left $\mathscr{A}$-module $\mathscr{A}$. Denote by W_i the simple $\mathscr{A}$-module V_i/V_{i-1}. Its commutant $\mathrm{End}_{\mathscr{A}}(W_i)$ is a division algebra ; since K is algebraically closed, we have $\mathrm{End}_{\mathscr{A}}(W_i) = K$. By Burnside's theorem ([B], § 4, n° 3, corollaire 1), the morphism φ_i from A to the bicommutant $\mathrm{End}_K(W_i)$ is surjective. It follows from 2.2 that the image in $\mathrm{End}_K(W_i)$ of the universal element α is the universal element of $\mathrm{End}_K(W_i)$, and hence its minimum polynomial χ_i is irreducible. If we choose a K-basis of $\mathscr{A}$ adapted to the composition series V_i, then the regular left representation of $\mathscr{A}$ provides an embedding $\mathscr{A} \hookrightarrow \mathrm{End}_K(\mathscr{A})$ whose image is block-triangular, with the i-th diagonal block isomorphic to $\mathrm{End}_K(W_i)$. By the Cayley-Hamilton theorem and the computation of a triangular determinant, the image of α in $\mathrm{End}_K(\mathscr{A})$ is cancelled by the product $\chi_1 \dots \chi_n$. It follows that P is a product of some of the χ_i's, say the first h, so that $\det = \det_1 \dots \det_h$. The facts that det is multiplicative and respects the unit follow.

We now prove (2). Let $Q(T) = \det(T - \alpha)$; it remains to prove that $Q(T) = P(T)$. For this it is enough to prove that Q cancels α. We consider the algebra $\mathscr{A} \otimes_R R'[T]$ and the element $T - \alpha$ therein. If we write $P(T) = (-1)^d T P_0(T) + P(0)$ where P_0 has degree $d-1$, the equality $P(\alpha) = 0$ gives $\alpha P_0(\alpha) = \det(\alpha)$. Since α is the universal element, we may look at this equality in $\mathscr{A} \otimes_R R'[T]$ and substitute $T - \alpha$ to α. Thus :

$$(T - \alpha) \cdot P_0(T - \alpha) = \det(T - \alpha) = Q(T) \; ,$$

Writing $P_0(T-\alpha) = T^{d-1} + \pi_1 T^{d-2} + \cdots + \pi_{d-1}$ and $Q(T) = q_0 T^d + \cdots + q_d$ with $\pi_i, q_j \in \mathscr{A} \otimes_R R'$, the above equality yields $\pi_0 = q_0$, $\pi_1 - \alpha\pi_0 = q_1$, ..., $-\alpha\pi_{d-1} = q_d$. Now multiply the first equality by 1, the second by α, ..., the last by α^d and add, we find $Q(\alpha) = 0$.

It remains to prove (3), but this is obvious from the universal formula $\alpha P_0(\alpha) = \det(\alpha)$. $\square$

Proposition 4.1.2 *Let $\mathfrak{A}$ be the universal n-dimensional algebra of degree d over the normalized stratum $\mathfrak{Alg}^{\sim}_{n,d}$. Then $\mathfrak{A}$ has a determinant satifying the properties (1) to (4) of theorem 4.1.1, and its formation commutes with any base change.*

Proof : The algebraic stack $\mathfrak{Alg}^{\sim}_{n,d}$ has an atlas $\mathrm{Alg}^{\sim}_{n,d} \to \mathfrak{Alg}^{\sim}_{n,d}$ which is smooth and affine, hence the fibre square of $\mathrm{Alg}^{\sim}_{n,d}$ over $\mathfrak{Alg}^{\sim}_{n,d}$ is a normal affine scheme, denoted $\mathrm{Alg}^{(2)}$. By theorem 4.1.1 there exists a determinant for $\mathfrak{A}$ after base extension to $\mathrm{Alg}^{\sim}_{n,d}$. The pullbacks of this determinant via the two projections $\mathrm{Alg}^{(2)} \to \mathrm{Alg}^{\sim}_{n,d}$ are two functions satisfying all the properties of theorem 4.1.1, hence by unicity they coincide, and it follows that the determinant descends to $\mathfrak{Alg}^{\sim}_{n,d}$. $\square$

It follows from this proposition that for any S-algebra scheme A such that the corresponding classifying morphism $S \to \mathfrak{Alg}_n$ factors through $f\colon S \to \mathfrak{Alg}^{\sim}_{n,d}$, we can define $\det_A := f^* \det_{\mathfrak{A}}$. When S is the spectrum of a normal domain, this is the same as the determinant given by theorem 4.1.1.

But of course, we would like more. The assumption of normality is used in one single place, in the beginning of the existence part of the proof. Thus, it is reasonable to expect that theorem 4.1.1 extends to an integral base scheme S. Then we may pass to $(\mathfrak{Alg}_{n,d})_{\mathrm{red}}$ by working on the irreducible components and then glue. So we ask :

Question 4.1.3 Can one construct a determinant on the reduced substack $(\mathfrak{Alg}_{n,d})_{\mathrm{red}}$?

4.2 Properties of the determinant

The determinant satisfies a strong invariance property with respect to automorphisms :

Proposition 4.2.1 *Under the assumptions of theorem 4.1.1, let $f\colon A \to A$ be a morphism of S-schemes that is either a ring scheme automorphism or a ring scheme antiautomorphism, and takes the scalars to scalars (i.e. $f(\mathbb{G}_{a,S}) = \mathbb{G}_{a,S}$). Then $\det_{A/S} \circ f = f \circ \det_{A/S}$. In particular if f is a $\mathbb{G}_{a,S}$-algebra (anti)automorphism then $\det_{A/S} \circ f = \det_{A/S}$.*

Proof : Let α be the universal element and let $P(T) = \det_{A/S}(T-\alpha)$ be its minimum polynomial. If we let f act on the polynomials by $f(T) = T$ and its natural action on the coefficients, since it is a ring (anti)automorphism we find that fP is the minimum polynomial of $f(\alpha)$. Moreover, $f(\alpha)$ is also a universal element, so it is uniquely a pullback of α, indeed, via f. Accordingly fP is the pullback f^*P. Looking at the constant term, we get the result. $\square$

Here is another useful property, which for simplicity we state in the affine case :

Proposition 4.2.2 *Let R be a normal domain and $\mathscr{A}$ an n-dimensional R-algebra. If the determinant $\det_{\mathscr{A}/R}$ is irreducible, then its degree d divides n.*

Proof : Let α be the universal element and let $P(T) = \det_{\mathscr{A}/S}(T-\alpha)$ be its minimum polynomial. If we extend the natural degree of $R_{\mathscr{A}} = \mathrm{Sym}(\mathscr{A}^\vee)$ by assigning degree 1 to T, then P is homogeneous of degree d. So if $P = QR$ in $R_{\mathscr{A}}[T]$, then Q and R are homogeneous of degrees e, f such that $d = e + f$. Since we assumed that $\det_{\mathscr{A}/R} = (-1)^d Q(0) R(0)$ is irreducible, it follows that either $Q(0) \in A^\times$ and $e = 0$, or $R(0) \in A^\times$ and $f = 0$. Therefore P is irreducible in $R_{\mathscr{A}}[T]$. Moreover the Cayley-Hamilton polynomial of α has the same irreducible factors as P, so by irreducibility $\mathrm{CH}_\alpha = P^m$ for some m. By taking degrees we get $n = dm$. $\square$

5 Computations and applications

5.1 More determinants

We start with the determinant of the opposite algebra, and the determinant of a product.

Proposition 5.1.1 *Under the assumptions of theorem 4.1.1, let $A^\circ \to S$ be the opposite algebra whose multiplication is the opposite as that of A, i.e. $a \star_{A^\circ} b = b \star_A a$. Then $\det_{A^\circ/S} = \det_{A/S}$.*

Proof : We have $A^\circ = A$ as vector bundles, so $\mathrm{Sym}^d(A^{\circ\vee}) = \mathrm{Sym}^d(A^\vee)$. It is clear that $\det_{A/S}$ satisfies the properties of theorem 4.1.1 for A°, hence by unicity $\det_{A^\circ/S} = \det_{A/S}$. $\square$

Proposition 5.1.2 *Let S be a normal integral scheme.*

(1) *If $f\colon A \to B$ a surjective homomorphism of finite-dimensional $\mathbb{G}_{a,S}$-algebra schemes with determinants $\det_{A/S} \in \mathrm{Sym}^d(A^\vee)$ and $\det_{B/S} \in \mathrm{Sym}^e(B^\vee)$, then there exists a unique section $\det_{A/B/S}$ of $\mathrm{Sym}^{d-e}(A^\vee)$ such that $\det_{A/S} = \det_{A/B/S} \cdot f^* \det_{B/S}$. Moreover $\det_{A/B/S}\colon A \to \mathbb{G}_{a,S}$ is a morphism of multiplicative unitary monoid schemes.*

(2) *If $C = A \times B$ is a product algebra, then $\det_{C/S} = \mathrm{pr}_1^* \det_{A/S} \cdot \mathrm{pr}_2^* \det_{B/S}$.*

Proof : (1) We can work locally over the base and hence suppose that $S = \mathrm{Spec}(R)$ is affine. Consider the R-algebra $\mathscr{A} = A(R)$, the function ring $R_{\mathscr{A}} = \mathrm{Sym}(\mathscr{A}^\vee)$, its fraction field $K_{\mathscr{A}} = \mathrm{Frac}(R_{\mathscr{A}})$, and the universal element α. Similarly we have $\mathscr{B}$, $R_{\mathscr{B}}$, $K_{\mathscr{B}}$, β.

Let $P \in R_{\mathscr{A}}[T]$ and $Q \in R_{\mathscr{B}}[T]$ be the minimum polynomials of α and β. The relation $(f \times \mathrm{id}_A) \circ \alpha = f^*\beta$ (see 2.2.1) applied to $P(\alpha) = 0$ gives $P(\beta) = 0$. Since f is a surjective map of vector bundles, it is dominant ; in particular $f^*\colon R_{\mathscr{B}} \to R_{\mathscr{A}}$ is injective. It follows that Q divides P in $K_{\mathscr{A}}[T]$, that is to say $P = QR$ for some $R \in K_{\mathscr{A}}[T]$. In fact $R \in R_{\mathscr{A}}[T]$ since Q is unitary. We define $\det_{A/B/S} := (-1)^{d-e} R(0)$. It is clear that it satisfies all the properties asserted in the statement of the proposition.
(2) If $C = A \times B$, we have projections $\mathrm{pr}_1\colon C \to A$ and $\mathrm{pr}_2\colon C \to B$. We localize and suppose that $S = \mathrm{Spec}(R)$ is small enough so that A and B are trivial as vector bundles. We end up with $\mathscr{C} = \mathscr{A} \times \mathscr{B}$, $R_{\mathscr{C}} = R_{\mathscr{A}} \otimes R_{\mathscr{B}}$, and the fraction field $K_{\mathscr{C}}$. With a slight abuse of notation we can write the universal element of C as $\gamma = (\alpha, \beta)$, if we think of the injections pr_1^* and pr_2^* as inclusions. From the above, the minimum polynomial of γ is a multiple of P and Q. We now argue that P and Q are coprime in $K_{\mathscr{C}}[T]$. For otherwise, the resultant $\mathrm{Res}(P, Q)$ vanishes, giving a relation of algebraic dependance between the variables of $R_{\mathscr{A}}$ and $R_{\mathscr{B}}$. Since $R_{\mathscr{A}}$ and $R_{\mathscr{B}}$ are polynomial rings in independent variables, this relation can involve only elements from R, hence P and Q belong to $R[T]$. But this is impossible since $d \geq 1$. Therefore P and Q are coprime, so the minimum polynomial of γ is PQ, and $\det_{C/S} = \det_{A/S} \det_{B/S}$. $\square$

It is natural to ask what is the dual result, namely, what is the determinant of a coproduct algebra. However the coproduct in the category of algebras is the free product, which is *not* finite-dimensional. Still, one can wish to compute the determinant of a tensor product $A \otimes B$, which is universal for pairs of maps $f : A \to C$, $g : B \to C$ whose images $f(A)$ and $g(B)$ commute. If $f\colon A \to B$ is an injective homomorphism of algebra schemes over a normal base, then it is not hard to prove that $\det_A$ divides $\det_B|_A$. In the case of a tensor product we get that $\det_A$ divides $\det_{A\otimes B}|_A$ and $\det_B$ divides $\det_{A\otimes B}|_B$, but it is not obvious how to guess an expression for $\det_{A\otimes B}$. For example, in view of the isomorphism $\mathrm{M}_p(k) \otimes_k \mathrm{M}_q(k) \simeq \mathrm{M}_{pq}(k)$, the formula we are looking for should give the determinant of (pq, pq) matrices in terms of the determinant of (p, p) and (q, q) matrices. To sum up :

Question 5.1.3 Is there a simple "formula" for the determinant of a tensor product $A \otimes B$ in terms of $\det_A$ and $\det_B$?

5.2 Traces and discriminants

Let S be a normal integral scheme, A an n-dimensional $\mathbb{G}_{a,S}$-algebra scheme, $\det_A$ its determinant of degree d, and α the universal element. We define some invariants of the intrinsic structure of A.

5.2.1 *Coefficients of the characteristic polynomial.* They are the sections c_i of $\mathrm{Sym}^i(A^\vee)$ defined by $\det_A(T-\alpha) = T^d - c_1T^{d-1} + \cdots + (-1)^d c_d$. The coefficient c_1 is called the *trace* and denoted tr_A. From theorem 4.1.1 it follows that the formation of the c_i commutes with flat extensions of normal integral schemes.

5.2.2 *Discriminant.* Locally over S we may choose a basis $e_1, \ldots, e_n$ for A and compute the determinant of the matrix whose (i,j)-th element is $\mathrm{tr}_A(e_ie_j)$. By the usual formula, a base change multiplies this by the square of the determinant of the transition matrix. Therefore we obtain a section of the quotient (monoid) stack $[\mathbb{G}_{a,S}/\mathbb{G}_{m,S}]$, where the multiplicative group $\mathbb{G}_{m,S}$ acts on $\mathbb{G}_{a,S}$ by the rule $z.x = z^2x$. These local sections are canonical and hence glue to a section over all of S. This is called the *discriminant* of A and denoted $\mathrm{disc}(A)$. Here again, the formation of $\mathrm{disc}(A)$ commutes with flat extensions of normal integral schemes.

5.2.3 *Unimodular group.* The *unimodular group* of A/S is the kernel of the determinant, namely $U = (\det_A)^{-1}(1)$. This is a subgroup scheme of the group $G = A^\times = (\det_A)^{-1}(\mathbb{G}_{m,S})$ of invertible elements of A.

5.3 Application to the topology of $\mathfrak{Alg}_n$

In this subsection, the base ring is a field k, and the moduli stack $\mathfrak{Alg}_n$ is considered over k.

Proposition 5.3.1 $\mathrm{Alg}_{n,n}$ *is irreducible of dimension* n^2*, so* $\overline{\mathrm{Alg}}_{n,n}$ *is an irreducible component of* Alg_n*. In other words,* $\mathrm{Alg}_{n,\leq n-1}$ *is a union of irreducible components of* Alg_n*.*

Proof : By lemma 3.2.1, $\mathfrak{Alg}_{n,n}$ is the open substack of algebras that are locally (for the étale topology) monogenic, hence commutative. Therefore the result follows from [P], § 6. The argument is so simple that we recall it shortly : the locus of étale algebras $E \subset \mathfrak{Alg}_{n,n}$ is defined by the nonvanishing of the discriminant, hence it is open and dense. Since an étale algebra splits after an étale base extension, the orbit of the split algebra $A = k^n$ under GL_n acting by base change is E. It follows that E is irreducible, hence so is $\mathfrak{Alg}_{n,n}$. □

In [P], questions 9.8-9.9, B. Poonen asks what is the functor of points of $\overline{\mathrm{Alg}}_{n,n}$ (his notation for $\overline{\mathrm{Alg}}_{n,n}$ is $\mathfrak{B}_n^{\mathrm{et}}$). We have no complete answer, but our result indicates that all monogenic algebras are points of this functor.

Now here is an application of the determinant to the topology of $\mathfrak{Alg}_n$:

Proposition 5.3.2 *Let* $n_1, \ldots, n_r \geq 1$ *be integers and* $n = (n_1)^2 + \cdots + (n_r)^2$*. Assume that one of the* n_i *is at least* 2 *and denote by* ν *their infimum. Then the irreducible component of the algebra* $\mathscr{A}_0 = \mathrm{M}_{n_1}(k) \times \cdots \times \mathrm{M}_{n_r}(k)$ *is contained in* $\mathfrak{Alg}_{n,\leq n-\nu}$*.*

Proof : Assume that the irreducible component of $\mathscr{A}_0$ meets $\mathfrak{Alg}_{n,d}$. Then there is a discrete valuation ring R with fraction field K and residue field k, and an R-algebra $\mathscr{A}$ such that $\mathscr{A} \otimes k \simeq \mathscr{A}_0$ and $\mathscr{A} \otimes K$ has degree d. Let α (resp. α_0) denote the universal element of $\mathscr{A}$ (resp. $\mathscr{A}_0$). If we had $d = n$, then $\mathscr{A} \otimes K$ would be commutative by lemma 3.2.1, and then $\mathscr{A}_0$ would be commutative also. Since this is not the case, we have $d < n$. We have a factorization of the Cayley-Hamilton polynomial of α as $\mathrm{CH}_\alpha(T) = \psi(T) \det_{\mathscr{A}}(T-\alpha)$, where the degree of ψ is $n - d > 0$. There is an analogous factorization for the Cayley-Hamilton polynomial of α_0, which is just the reduction of $\mathrm{CH}_\alpha(T)$ modulo the maximal ideal of R. Let δ_i be the determinant of the algebra $\mathrm{M}_{n_i}(k)$, irreducible of degree n_i. We know from

proposition 5.1.2 that $\det_{\mathscr{A}_0}(T-\alpha_0)$ is the product of the δ_i. Moreover $\mathrm{CH}_{\alpha_0}(T)$ and $\det_{\mathscr{A}_0}(T-\alpha_0)$ have the same irreducible factors, hence $\overline{\psi}(T)$ is divisible by one of the δ_i. Therefore its degree is at least ν, that is, $n-d\geq\nu$. So $d\leq n-\nu$. □

These results raise the natural question :

Question 5.3.3 Is $\mathfrak{Alg}_{n,\leq d}$ (or $\mathrm{Alg}_{n,\leq d}$) a union of irreducible components of $\mathfrak{Alg}_n$ (or Alg_n) ?

6 Examples

6.1 Determinants of some classical algebras

(1) Let K/k be a finite Galois field extension with Galois group G. Then $\det_{K/k}$ is the norm defined by $N(x)=\prod_{g\in G} g(x)$.

(2) Let $k=k_0(t_1,\dots,t_s)$ be the field of rational functions in $t_1,\dots,t_s$ over a field of characteristic $p>0$ and let K/k be the finite purely inseparable extension generated by s elements $u_i=(t_i)^{1/p}$. A natural basis of K/k is given by the monomials $u^I:=(u_1)^{i_1}\dots(u_s)^{i_s}$ for $0\leq i_1,\dots,i_s\leq p-1$. For $x=\sum_I x_I u^I$ one finds $\det_{K/k}(x)=-\sum_I (x_I)^p t^I$. (Note that $x^p=\sum_I (x_I)^p t^I$).

(3) Let $H=H_{\alpha,\beta}$ be the quaternion algebra generated by $1,i,j,k$ with relations $i^2=\alpha$, $j^2=\beta$, $ij=-ji=k$. Then $\det_{H/k}$ is the norm $N(a+bi+cj+dk)=a^2-\alpha b^2-\beta c^2+\alpha\beta d^2$.

(4) Let $A=\wedge^* E$ be the exterior algebra of an r-dimensional vector space E. To any basis $\{e_1,\dots,e_n\}$ of E is associated a basis of A whose I-th vector is $e_I=e_{i_1}\wedge\dots\wedge e_{i_k}$, for all $0\leq k\leq n$ and $I=\{i_1<\dots<i_k\}\subset\{1,\dots,n\}$. Then $\det_{A/k}(x)=(x_\emptyset)^r$, where $x=\sum x_I e_I$.

It would be interesting to generalize examples (3) and (4) by computing the determinant of a general Clifford algebra.

6.2 Algebras of dimension 3

The simplest case is dimension $n=2$, but then the determinant is just the Cayley-Hamilton polynomial. Precisely, for a 2-dimensional algebra endowed with a basis $\{1,x\}$, the datum of an equation $x^2=ax+b$ determines a multiplication that is automatically associative, thus Alg_2 is an affine plane. One checks that $\det(r+sx)=r^2-s^2b+ars$.

Next there is the case of dimension $n=3$, which is still easily tractable. In [MT], Miranda and Teicher study the noncommutative 3-dimensional algebras over integral schemes. It is in fact not more complicated to decribe directly the whole moduli stack $\mathfrak{Alg}_3$. As we will see, the scheme Alg_3 has two irreducible components : one is the component of commutative algebras, which is also the closure of $\mathrm{Alg}_{3,3}$, and the other is just $\mathrm{Alg}_{3,\leq 2}=\mathrm{Alg}_{3,2}$. We will pay particular attention to $\mathrm{Alg}_{3,2}$, by describing the base change group action and giving the determinant on $\mathrm{Alg}_{3,2}$.

6.2.1 *Equations for* Alg_3. The multiplication table of a 3-dimensional algebra with basis $\{1,x,y\}$ looks like this :

$$\begin{aligned} x^2 &= a+bx+cy \\ xy &= d+ex+fy \\ yx &= g+hx+iy \\ y^2 &= j+kx+ly \end{aligned}$$

The conditions of associativity are given by the following eight sets of equations :

$(x^2)x=x(x^2)$
$c(d-g)=0$
$c(e-h)=0$
$c(f-i)=0$

$(x^2)y=x(xy)$
$bd+cj=ea+fd$
$be+ck=d+eb+fe$
$a+bf+cl=ec+f^2$

$(xy)x = x(yx)$
$ea + fg = ha + id$
$d + eb + fh = g + hb + ie$
$ec + fi = hc + if$

$(xy)y = x(y^2)$
$ed + fj = ka + ld$
$e^2 + fk = j + kb + le$
$d + ef + lf = kc + lf$

$(yx)x = y(x^2)$
$ha + ig = bg + cj$
$g + hb + ih = bh + ck$
$hc + i^2 = a + bi + cl$

$(yx)y = y(xy)$
$hd + ij = eg + fj$
$he + ik = eh + fk$
$g + hf + il = d + ei + fl$

$(y^2)x = y(yx)$
$ka + lg = hg + ij$
$j + kb + lh = h^2 + ik$
$kc + li = g + hi + il$

$(y^2)y = y(y^2)$
$k(d - g) = 0$
$k(e - h) = 0$
$k(f - i) = 0$

These twenty-four equations can be rearranged and simplified to give the following twelve :

$$\begin{cases} a = f(f-b) + c(e-l) \\ d = ck - ef \\ g = ck - hi \\ j = k(f-b) + e(e-l) \end{cases} \quad \text{and} \quad \begin{cases} c(e-h) = c(f-i) = 0 \\ k(e-h) = k(f-i) = 0 \\ (f+i-b)(e-h) = (f+i-b)(f-i) = 0 \\ (e+h-l)(e-h) = (e+h-l)(f-i) = 0 \end{cases}$$

The first four equations imply that Alg_3 is a subscheme of affine 8-space in coordinates b, c, e, f, h, i, k, l. The remaining eight equations generate an ideal which is the product of the two prime ideals $p_1 = (e-h, f-i)$ and $p_2 = (c, k, f+i-b, e+h-l)$. So Alg_3 is defined in $\mathbb{A}^8$ by the ideal p_1p_2, it has two irreducible components which are affine spaces.

6.2.2 *The component of commutative algebras.* The component defined by $p_1 = 0$ is the component of commutative algebras ; it is also the closure of $\mathrm{Alg}_{3,3}$; it is isomorphic to affine 6-space with coordinates b, c, e, f, k, l. The multiplication table of an algebra in this component is the following :

$$\begin{array}{rcl} x^2 & = & f(f-b) + c(e-l) + bx + cy \\ xy = yx & = & ck - ef + ex + fy \\ y^2 & = & k(f-b) + e(e-l) + kx + ly \end{array}$$

6.2.3 *The component of degree 2 algebras.* We now pay more attention to the component defined by $p_2 = 0$, and we start by checking that it is $\mathrm{Alg}_{3,\le 2} = \mathrm{Alg}_{3,2}$. Let $\alpha = r + sx + ty$ be the universal element, we have

$$\alpha^2 = \big[r^2 + st(d+g) + s^2a + t^2j\big] + \big[2rs + st(e+h) + s^2b + t^2k\big]x + \big[2rt + st(f+i) + s^2c + t^2l\big]y\ .$$

The locus $\mathrm{Alg}_{3,\le 2}$ is defined by the vanishing of $1 \wedge \alpha \wedge \alpha^2$, hence :

$$\begin{vmatrix} 2rs + st(e+h) + s^2b + t^2k & s \\ 2rt + st(f+i) + s^2c + t^2l & t \end{vmatrix} = -cs^3 + (b-f-i)s^2t + (e+h-l)st^2 + kt^3 = 0$$

as polynomials in s, t. We find $b = f + i$, $c = k = 0$, $l = e + h$, in other words this is indeed the component defined by $p_2 = 0$. Thus $\mathrm{Alg}_{3,2}$ is isomorphic to affine 4-space in coordinates e, f, h, i. The multiplication table of an algebra in this component is the following :

$$\begin{array}{rcl} x^2 & = & -fi + (f+i)x \\ xy & = & -ef + ex + fy \\ yx & = & -hi + hx + iy \\ y^2 & = & -eh + (e+h)y \end{array}$$

The minimum polynomial of α is $P(T) = (T - (r + is + et))\,(T - (r + fs + ht))$ and the determinant is $\det(\alpha) = (r + is + et)(r + fs + ht)$. Note that by proposition 4.2.2, we expected it to split.

We shortly describe $\mathfrak{Alg}_{3,2}$. Base change $x' = r + sx + ty$, $y' = u + vx + wy$ is encoded by a transition matrix

$$\begin{pmatrix} 1 & r & u \\ 0 & s & v \\ 0 & t & w \end{pmatrix}$$

Set $\delta = sw - tv$ so that $x = \frac{tu-rw}{\delta} + \frac{w}{\delta}x' - \frac{t}{\delta}y'$ and $y = \frac{vr-su}{\delta} - \frac{v}{\delta}x' + \frac{s}{\delta}y'$. Then one can calculate $(x')^2$, $x'y'$, $y'x'$, $(y')^2$ in terms of x', y' and after a (tedious) computation one finds :

$$\begin{aligned} e' &= u + vi + we \\ f' &= r + sf + th \\ h' &= u + vf + wh \\ i' &= r + si + te\ . \end{aligned}$$

The stabilizer of a point of $\mathrm{Alg}_{3,2}$ with coordinates (e, f, h, i) has equations

$$\begin{aligned} &u + vi + (w-1)e = 0 \\ &r + (s-1)f + th = 0 \\ &u + vf + (w-1)h = 0 \\ &r + (s-1)i + te = 0\ , \end{aligned}$$

or in other words $v(f - i) = (w - 1)(e - h)$, $(s - 1)(f - i) = t(e - h)$, $u + vi + (w - 1)e = 0$, $r + (s - 1)f + th = 0$. Let F be the 2-dimensional intersection of $\mathrm{Alg}_{3,2}$ with the other irreducible component and $U = \mathrm{Alg}_{3,2} \setminus F$ be the complement. We see that for points in F, the stabilizer has dimension 4. The image of F in $\mathfrak{Alg}_{3,2}$ is a single point which is the unique commutative algebra, isomorphic to $k[x, y]/(x^2, xy, y^2)$, with automorphism group $\mathrm{GL}_2(k)$. Away from F, the stabilizer has dimension 2 and the image of U in $\mathfrak{Alg}_{3,2}$ has dimension 2, composed of algebras with 2-dimensional automorphism group.

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Institut de Mathématiques
Théorie des Nombres
Université Pierre et Marie Curie
Case 82
4, place Jussieu
F-75252 Paris Cedex 05
romagny@math.jussieu.fr